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which is a necessary condition to apply the time-sharing argument to an AVC subject to a state constraint.

The fact that (1) does not imply (16) also appears to prevent one from applying Ahlswede's elimination technique [1], [7], to AVC's subject to a state constraint; thus, our result in Theorem 1 does not appear to be sufficient to allow us to prove the conjecture that the region in (2) is exactly the capacity region when state constraints are present, either by the elimination technique or by time-sharing.

Corrections to [6]

The preceding observations require us to make the following corrections to our prior paper [6].

In [6, Theorem 5.8] the words "closed convex hull," should be replaced by "closure."

In [6, Section V-C] we can no longer conclude that equation (5.8) is correct. Instead, all that we can conclude, using notation defined in [6], is that the closure of

$$\mathscr{R}_X^{1/2}(p^*, q^*, W_a) \cup \mathscr{R}_Y^{1/2}(p^*, q^*, W_a)$$

is a subset of $C(W_a, 1/2)$ and that $C(W_a, 1/2)$ is a subset of the right-hand side of (5.8).

ACKNOWLEDGMENT

The author is grateful to an anonymous reviewer of an earlier version of this correspondence for reminding him that the time-sharing argument can not be applied because (1) does not imply (16).

APPENDIX

Lemma: Let p be any distribution with $p(x) > 0$ for all $x \in \mathrm{G}$. For any $q \in \mathscr{D}(\mathrm{G})$, $H(p*q) - H(q) = 0$, if and only if q is the uniform distribution.

Proof: Define a distribution on $\mathrm{G} \times \mathrm{G}$ by setting $P_{XY}(x, y) = p(x)q(y-x)$. Then

$$\begin{aligned} I(X \wedge Y) &= H(Y) - H(Y \mid X) \\ &= H(p*q) - H(q). \end{aligned}$$

Clearly, $H(p*q) - H(q) = 0$, if and only if X and Y are independent. Since $p(x) > 0$ for all x, X and Y are independent, if and only if $q(y-x) = q(y)$ for all $x, y \in \mathrm{G}$. Thus, X and Y are independent, if and only if q is uniform. □

Theorem A: Let W be a general multiple-access AVC. If $p \in \mathscr{D}(\mathrm{X})$ and $q \in \mathscr{D}(\mathrm{Y})$ are such that (recall paragraph 2 of Section III)

$$I(X \wedge Z) > I(S \wedge Z), \quad \text{for all } r \in \mathscr{D}^L(\mathrm{S}), \quad (\mathrm{A}.1)$$

and

$$I(Y \wedge Z \mid X) > I(S \wedge Z \mid X), \quad \text{for all } r \in \mathscr{D}^L(\mathrm{S}), \quad (\mathrm{A}.2)$$

then every pair (R_1, R_2) satisfying

$$0 < R_1 < I^L(X \wedge Z) \quad \text{and} \quad 0 < R_2 < I^L(Y \wedge Z \mid X) \quad (\mathrm{A}.3)$$

belongs to the deterministic-code average-probability-of-error capacity region under state constraint L.

Proof: This theorem can be proved by making trivial modifications to the proof of [5, Theorem 5.5]. A similar observation was made in [6, Section V-A], though it was not pointed out there that in this case the modifications do not require that the channel be nonsymmetrizable [6, Definitions 3.3, 3.5, and 3.7]. □

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Shaping Using Variable-Size Regions

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Abstract—**Constellation shaping is extended to provide shaping gains without resorting to high-dimensional constellations. This is accomplished by dividing the constellation into unequal sized constellations, and selecting these constellations on an equiprobable basis. A design example is provided, demonstrating the simplicity and power of the approach.**

Index Terms—**Coding, modulation, shaping, signal constellations, and nonequiprobable signaling.**

I. INTRODUCTION

Coding schemes for transmission of data over the Gaussian channel have been used over the last decade that lead to improved performance [1]–[4]. The most popular approach is to use *coset codes*, and to attain high code rates coupled with good performance, the shift has been to use higher dimensional constellations. One result of moving to higher dimensions is the ability to achieve what has been called *shaping gains*. This is due to the reduction in average symbol energy that can accompany the use of a constellation whose boundary is not an N-cube. In particular, as the constellation becomes more spherical, it enforces a nonequiprobable distribution on signal points drawn from a constituent two-dimensional constellation. It has been shown [5] that in the limit as $N \to \infty$, an N-sphere can achieve 1.53 dB of shaping gain, and will enforce a truncated Gaussian distribution on the constituent 2-D constellation. Attention has been focused on these shaping gains, as they can be achieved independently of any gain due to the use of a coset code.

Multidimensional constellations with significant shaping gain were first described by Conway and Sloane in [6]. Other constellations with shaping gain and simple decoding methods were described by

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Calderbank and Sloane in [7]. Forney [8] extended the method in [7] to more general lattice partitions. Subsequently, Forney has applied this approach to constellations constructed from trellis codes in [9], and Eyuboglu and Forney have applied it to nonideal channels [10].

Recently, it has been proposed that the most direct route to achieve shaping gains is to use lower dimensional constellations and enforce a nonequiprobable distribution on the signal points [11]. Using this method, Calderbank and Ozarow have been able to equal the best shaping gains found using higher dimensional constellations, and yet have better peak-to-average power ratios (PAR) than found in [8].

The method of Calderbank and Ozarow divides the constellation into T subconstellations of equal size. A *shaping code* is then applied to select subconstellations according to a given probability distribution. Signal points within subconstellations are equiprobable.

In this correspondence, we consider extensions of the work of Calderbank and Ozarow. In Section II, we describe shaping as proposed by Calderbank and Ozarow, and derive an alternate method of shaping that offers higher shaping gains. We also extend the PAR calculations to this new approach. This method is essentially the same as that first described in [2], but found through a different route. In Section III, we generalize the results of Section II, and describe our shaping method as the dual of source coding using prefix codes. In Section IV, we compare shaping using prefix codes to previously published shaping gains, and demonstrate that higher shaping gains are possible with potentially little complexity. In Section V, we describe the major drawback of this method, namely that a random number of bits are now transmitted per channel use. We describe two methods proposed to alleviate the problems associated with this fact ([12], [13]), and offer an application where it may have little impact. Section VI provides an example of signaling using nonequal subconstellation sizes, and demonstrates that for very little complexity shaping gains of 1 dB can be achieved. Section VII summarizes the results.

II. Shaping Gains

Shaping gains refer to the reduction in average signal power by constellation design and signaling choice. In [11], the total shaping gain is contributed to by two separable factors. The first is the shaping gain of the constellation chosen. This gain varies between 0 dB (for N-cube) to 1.53 dB for a spherical constellation in an infinite number of dimensions. The second factor contributing to the shaping gain is what Calderbank and Ozarow call the *biasing gain*, and comes about through the use of a nonequiprobable signaling scheme.

A. Equal Sized Subconstellations

In [11], the nonequiprobable signaling is achieved by first dividing the constellation into T subconstellations, all of equal size. This subdivision is defined by first defining a fundamental region $\mathcal{R}$. The construction continues by scaling the fundamental region T times, to form a constellation as in Fig. 1. Thus, the T copies of the basic region are $\mathcal{R}$, $\alpha_1\mathcal{R}, \cdots, \alpha_{T-1}\mathcal{R}$. With the nested sequence of scaled regions defined in this manner, we may define the total constellation, Ω, as the union of T subconstellations, Ω_i, $i = 0, \cdots, T-1$, where $\Omega_i = \Omega \cap (\alpha_{i+1}\mathcal{R} / \alpha_i\mathcal{R})$. Signal points in each subconstellation are used equiprobably, and subconstellation Ω_i is used with frequency f_i.

In this correspondence, Calderbank and Ozarow make use of the continuous approximation to compute average power. This approximation assumes that the constellation consists of a set of continuous points. This gives results that are easily calculated, and yet are accurate for large constellations. Using this method, the biasing gain normalized to two dimensions is given by [11],

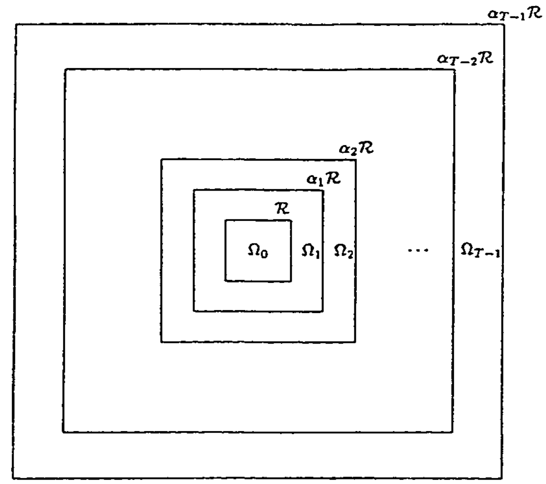

Fig. 1. Decomposition of a rectangular constellation into subconstellations.

$$\gamma_N(f_0, \cdots, f_{T-1}) = \frac{2^{(2/N)H(f_0, \cdots, f_{T-1})}}{\sum_{i=0}^{T-1}\left[(i+1)^{2/N+1} - i^{2/N+1}\right] f_i}, \quad (1)$$

where T is the number of subconstellations, N is the number of dimensions, and f_i is the probability that subconstellation i, $i = 0, \cdots, T-1$, will be chosen. The reader is referred to [11] for details.

In [11], the increase in the PAR due to constellation expansion and the use of a shaping code was found to be

$$\mathrm{PAR}_2 = \frac{T^{2/N}}{\sum_{i=0}^{T-1}\left[(i+1)^{2/N+1} - i^{2/N+1}\right] f_i}, \quad (2)$$

where T and the f_i are as in (1). Thus the peak-to-average power ratio of the nonequiprobable signaling constellation is given by

$$\mathrm{PAR}_\Omega = \mathrm{PAR}_2 \times \mathrm{PAR}_{\mathcal{R}}, \quad (3)$$

where $\mathrm{PAR}_{\mathcal{R}}$ is the peak-to-average power ratio of the fundamental region, $\mathcal{R}$, and PAR_Ω is the peak-to-average power ratio of the total constellation, Ω.

Calderbank and Ozarow are able to achieve their shaping gains without increasing the PAR beyond that found using higher dimensional constellations. In fact, they find that their method yields equal shaping gains, but with better PAR figures. In some cases, the improvement is by a factor of 2 or more over the other methods.

B. Nonequal Subconstellations

We are left to wonder if this construction is optimal, namely, the splitting of the region into T equal sized subconstellations. [11] argues that equal sized subconstellations are better than subconstellations with diminishing size, i.e., with $V(\Omega_0) > V(\Omega_1) > \cdots > V(\Omega_{T-1})$, where $V(\Omega_i)$ represents the volume of region Ω_i. They provide an example with $V(\Omega_i) = 2V(\Omega_{i+1})$, where this is indeed the case. However, other relations between the subconstellations are possible.

We begin by generalizing (1) to find the biasing gain for subconstellations that are not equal. We first note that in [11], the average

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transmitted signal power (normalized to two dimensions) is derived from the expression:

$$\frac{1}{(i+1)}\sum_{j=0}^{i} t_j P_0 = (i+1)^{2/N} P_0, \tag{4}$$

where P_0 is the average power of the fundamental constellation Ω_0. Recall also that the number of points in each subconstellation is equal.

Now let the subconstellations be constructed as follows. Again we scale the fundamental region T times, yielding T copies of the basic region $\mathcal{R}, \alpha_1 \mathcal{R}, \cdots, \alpha_{T-1}\mathcal{R}$ as before. However, now let the number of points in each subconstellation, Ω_i, $i = 0, \cdots, T-1$, vary. In fact, let us define

$$k_i = \frac{V(\alpha_i \mathcal{R})}{V(\mathcal{R})}. \tag{5}$$

The quantity $(k_j - k_{j-1})/k_{T-1}$ then represents the fraction of signal points in the jth subconstellation, and the following holds true:

$$\frac{1}{k_{T-1}}\sum_{j=0}^{T-1}(t_j P_0)(k_j - k_{j-1}) = k_{T-1}^{2/N} P_0. \tag{6}$$

Solving for t_j yields

$$t_j = \frac{k_j^{2/N+1} - k_{j-1}^{2/N+1}}{k_j - k_{j-1}}. \tag{7}$$

Thus, the average power in the jth annular subconstellation is

$$t_j P_0 = \frac{k_j^{2/N+1} - k_{j-1}^{2/N+1}}{k_j - k_{j-1}} P_0. \tag{8}$$

Given we choose subconstellation Ω_i with frequency f_i, then the total average power is

$$P = \left[\sum_{j=0}^{T-1} \frac{k_j^{2/N+1} - k_{j-1}^{2/N+1}}{k_j - k_{j-1}} f_i\right] P_0. \tag{9}$$

Selecting subconstellation j increases the rate by $\log_2(k_j - k_{j-1})$, thus, using nonequiprobable signaling with unequal subconstellations yields a total rate of

$$\left[\sum_{i=0}^{T-1} f_i \log(k_i - k_{i-1}) + H(f_0, \cdots, f_{T-1})\right] \tag{10}$$

bits per channel use. Equiprobable signaling at this rate yields an average power of

$$2^{2/N}\left[\sum_{i=0}^{T-1} f_i \log(k_i - k_{i-1}) + H(f_0, \cdots, f_{T-1})\right] P_0. \tag{11}$$

This yields a biasing gain normalized to two dimensions of

$$\gamma_N(f_0, \cdots, f_{T-1}) = \frac{2^{2/N}\left[\sum_{i=0}^{T-1} f_i \log(k_i - k_{i-1}) + H(f_0, \cdots, f_{T-1})\right]}{\sum_{i=0}^{T-1}\left[\frac{k_i^{2/N+1} - k_{i-1}^{2/N+1}}{k_i - k_{i-1}}\right] f_i}. \tag{12}$$

Continuing with the approach of using different sized subconstellations, we find that the increase in the PAR is given by

$$\mathrm{PAR}_2 = \frac{k_{T-1}^{2/N}}{\sum_{i=0}^{T-1}\left[\frac{k_i^{2/N+1} - k_{i-1}^{2/N+1}}{k_i - k_{i-1}}\right] f_i}, \tag{13}$$

where T, the f_i, and k_i are as in (12).

At this point we pause to note that using the new construction, it is theoretically possible to achieve slightly higher shaping gains and slightly better PAR's than using equal subconstellations. However, as will be seen in Section IV, the advantage of the new construction is to be found when using realizable shaping codes.

III. Generalization of Construction

We have been motivated so far by extending the method of Calderbank and Ozarow to nonequal subconstellation sizes. Through this approach, we have arrived at an expression that yields the shaping gain for a wide range of relationships between constellations. A more general description can be found, and in fact points out the relationship between this method and that described in [2].

In [2], the idea of shaping gains using prefix codes was described. This method can be explained as follows. If we desire to send an average of β bits per two-dimensions, then start with a two-dimensional constellation with $M > 2^{\beta}$ points, and assign a β_j-bit label to the jth point such that

$$\sum_j P_j \beta_j = \beta, \tag{14}$$

where P_j is the probability of selecting the jth point. If our selection of labels is done in such a way as to satisfy the Kraft inequality,

$$\sum_j 2^{-\beta_j} \le 1, \tag{15}$$

then there exists a prefix code of alphabet size 2 with the integers $\beta_1, \beta_2, \cdots$ as code-word lengths. If the signal points are selected by applying the prefix code to the incoming bit stream, and if the bits are equally likely, then the probability of selecting signal point j is now $2^{-\beta_j}$. If we assign longer labels to points farther from the origin, then a Gaussian distribution can be approximated. A simple way to implement this scheme would be to group points together (i.e., define a subconstellation as described in the previous section) and assign fixed-length labels to points in the same subconstellation. This description of shaping may be regarded as the dual to the use of prefix codes in source coding.

This formulation can be seen to be more general than that described in Section II. The simple implementation previously described may also be seen to include as a natural progression the case where $k_j - k_{j-1}$ decreases with increasing j, as with the generalized multidimensional cross constellations described by Forney and Wei ([5]), to $k_j - k_{j-1} = K$, a constant, as in Calderbank and Ozarow, to where $k_j - k_{j-1}$ increases with increasing j, as described in Section II. In fact, one may generalize the multidimensional cross constellation construction even further by splitting the constellation into more than two subconstellations. The construction then proceeds along the same lines as found in [5], and preserves the important property of transmitting an equal number of bits per multidimensional signal point. However, (12) may be seen as a simple way to calculate the biasing gain provided by the general approach if we assign fixed-length labels to points in the same subconstellation.

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IV. Implementation of Coding

A. Shaping Codes

Calderbank and Ozarow propose the use of a nonlinear code to provide the shaping. The code is defined as follows. Let $B_{n,m}$ be the set of all binary n-tuples v with Hamming weight $\mathrm{wt}(v) \le m$. Then, the rate of the code is given by

$$R(B_{n,m}) = \frac{1}{n} \log_2 \left(\sum_{j=0}^{m} \binom{n}{j} \right). \tag{16}$$

This code can be used to select two subconstellations with probability f_0 and f_1, by selecting constellation Ω_0 when a zero is present in the codeword, and selecting constellation Ω_1 when a one is present. The probability f_0 is given by

$$f_0 = \frac{\sum_{j=0}^{m} \binom{n}{j} (n-j)}{\left[\sum_{j=0}^{m} \binom{n}{j} \right] n}. \tag{17}$$

Note also that when using a code, the term $H(f_0, f_1, \cdots, f_{T-1})$ in (1) and (12) is replaced by $R(B_{n,m})$, the rate of the code.

Using these codes for $T = 2$, Calderbank and Ozarow are able to achieve practical biasing gains of 0.51 dB, using a $B_{20,5}$ nonlinear code. However, encoding such a code can be difficult. For more practical length codes, where a lookup table can be used, gains are less than half a dB. Table I lists some of these codes for $T = 2$ and $T = 4$, with associated code parameters and shape gains. Note that for $T = 4$, [11] combines two $B_{n,m}$ codes to select between the four subconstellations.

When we turn to nonequal subconstellations, we find that the best $B_{n,m}$ code for any T is a $B_{n,n}$ code. That is, we choose subconstellations equiprobably. This corresponds to the use of a rate $R = \log_2(T)$ code. If we were using equal sized subconstellations, then a rate $R = \log_2(T)$ code simply implements an equiprobable signaling scheme. However, with nonequal subconstellations, using equiprobable subconstellation selection still yields nonequiprobable signaling on the whole.

The shaping code now reduces to labeling the T subconstellations with binary n-tuples of length $\log_2(T)$. We use $\log_2(T)$ bits at the input of the coder to select the subconstellation, and signals are then selected from the subconstellations on an equiprobable basis. Given that the number of symbols in the jth subconstellation is M_j, then the total number of bits required to select any symbol in subconstellation j is $\log_2(M_j) + \log_2(T)$ bits. This fits in well with the description of shaping as the dual to source coding using a prefix code, as given in Section III. Here, $\beta_i = \log_2(M_j) + \log_2(T)$, if signal point i is in the jth subconstellation.

Since there are a different number of symbols in each subconstellation, a serial to parallel converter can be used to provide the bits required to select the symbol from the subconstellation. In Fig. 2, we have illustrated a block diagram that implements a shaping code.

Using $T = 2$, and splitting the two constellations into nonequal sizes, we find that the optimum ratio of the outer constellation size to the inner constellation size for equiprobable subconstellation selection is 3 to 1 (i.e., $k_1 = 4$, $k_0 = 1$). This yields a biasing gain of $2/\sqrt{3}$ (0.62 dB), a PAR_2 of 1.32, and a constellation expansion ratio of 1.15. (As an aside, the biasing gain is exactly the ratio of the density of the lattice A_2 to the density of lattice Z^2 ([7]).) Thus we can theoretically achieve a shaping gain of 0.82 dB at a cost of a 15 per-cent increase in constellation size, and a 32 per-cent increase in PAR. All of this is achievable without significant increase in complexity. Table II lists relevant parameters for $T = 2$ and $T = 4$

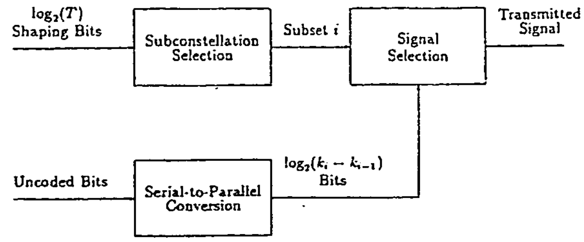

Fig. 2. Block diagram of shaping code implementation.

TABLE I
Biasing and Shaping Gains for Shaping Codes as Found in [11]

N	T	n	m_1	m_2	γ_2	CER_2	PAR_2	γ	PAR
2	2	6	2	—	0.35	1.19	1.29	0.55	2.58
2	2	12	3	—	0.45	1.24	1.38	0.65	2.76
2	2	18	4	—	0.51	1.26	1.41	0.71	2.82
2	4	6	1	3	0.43	1.55	1.71	0.63	3.42
2	4	12	2	5	0.60	1.50	1.73	0.80	3.46
2	4	18	3	7	0.68	1.48	1.73	0.88	3.46

equiprobable selection of unequal subconstellations. Note that these in all cases have better biasing gain and PAR_2 than the codes listed in Table I for the same T. In Table III, we list the shaping gains available using multidimensional equiprobable signaling for various lattices as described in [11]. The Leech lattice is denoted Λ_{24}, γ_s denotes the shaping gain of the Voronoi region of the lattice, and PAR_2 denotes the peak-to-average power ratio of the constituent 2-D constellation. Note that the equiprobable subconstellation approach also outperforms each of these lattices.

B. Realizable Shaping Code Constellations

One drawback to the use of non-equal subconstellations is the requirement (in binary signaling) that the number of signal points in each subconstellation must equal a power of two. Under this restriction, the k_i in (2) can take on only a discrete set of values. (This restriction can be relaxed somewhat at the expense of implementation complexity).

In Fig. 3, we have plotted the realizable biasing gain for $T = 2, 4, 8, 16$ versus the increase in PAR over the constituent constellation, PAR_2. For comparison, in Fig. 4 we have plotted biasing gain vs. PAR_2 for $T = 2, 4$ for both approaches. Note that the new method in all cases has both higher biasing gain and lower PAR_2. In both figures, (U) refers to the unequal subconstellation construction, and (E) refers to the equal subconstellation construction. In Fig. 5, we have plotted the maximum shape gain for both methods and several lattices vs. the peak-to-average power ratio. The parameters for the lattices are found in [11] and [8]. We assume a spherical 2-D constellation for the shaping codes. The shaping gain includes both the biasing gain and the constellation shaping gain. The PAR is the PAR of the constellation for methods (U) and (E), and is the PAR of the constituent 2-D constellation for the lattices. The reader is referred to [8] for an explanation of the notation for the lattices. Here, we see that the new method achieves higher gains at lower PAR than any of the other approaches. We also point out that the PAR of the lattices can be reduced by using peak constraints.

V. Encoding and Decoding Issues

Here, we briefly describe decoding of the shape codes. In [11], a level slicing approach is proposed, where the received signal is used to decide which subconstellation was in use. If a $B_{n,m}$ code is used,

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TABLE II
BIASING AND SHAPING GAINS FOR EQUIPROBABLE SELECTION OF SUBCONSTELLATIONS FOR $T = 2$ AND $T = 4$

N	T	$k1$	$k2$	$k3$	γ_2	CER_2	PAR_2	γ	PAR
2	2	4	—	—	0.62	1.15	1.32	0.82	2.64
2	4	2.4	4.7	11.7	0.97	1.34	1.68	1.17	3.35

TABLE III
SHAPING GAIN PARAMETERS OF SOME POPULAR LATTICES

N	Λ	$\gamma_s(\Lambda)$	$CER_s(\Lambda)$	$PAR_2(\Lambda)$
1	Z	0.00	1.0	3.0
4	D_4	0.37	1.41	4.62
8	D_8	0.40	1.68	5.54
8	E_8	0.65	2.0	6.98
24	Λ_{24}	1.03	4	15.2

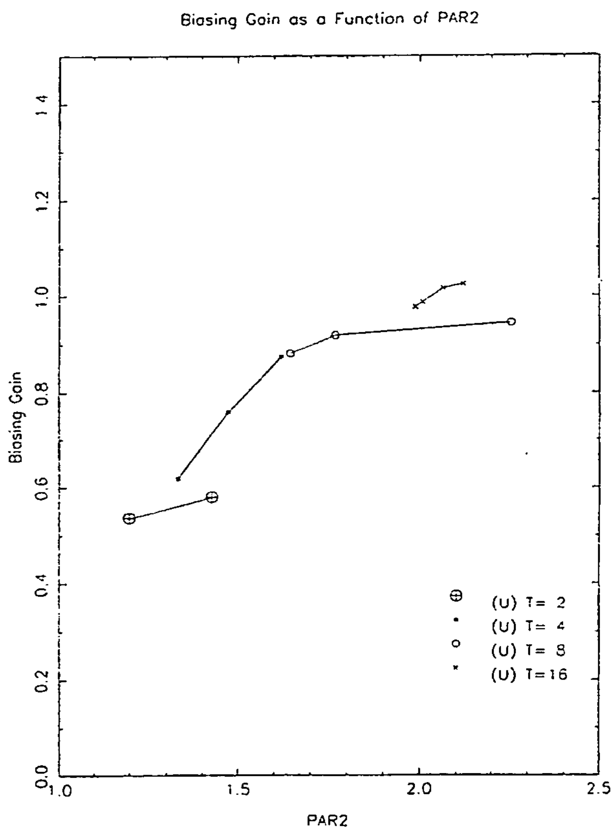

Fig. 3. Realizable biasing gains for new construction and $T = 2, 4, 8, 16$.

then after n symbols have been received and decoded by subconstellation, we can form the codeword used to provide shaping. This codeword is then used to find the actual data, by using table lookup, or some other block decoder method.

Since this method relies on the use of a nonlinear code, and the minimum distance of the code is one, this approach can lead to increased errors. For example, using $T = 2$, and a $B_{20,5}$ code, if one symbol is received in error causing us to select the wrong subconstellation, then with high probability we will decode incorrectly. This incorrect decoding can affect up to 14 bits at the output of the decoder. However, an important property of this approach is that each symbol transmitted represents the same number of bits, thus limiting their effect to a finite length block.

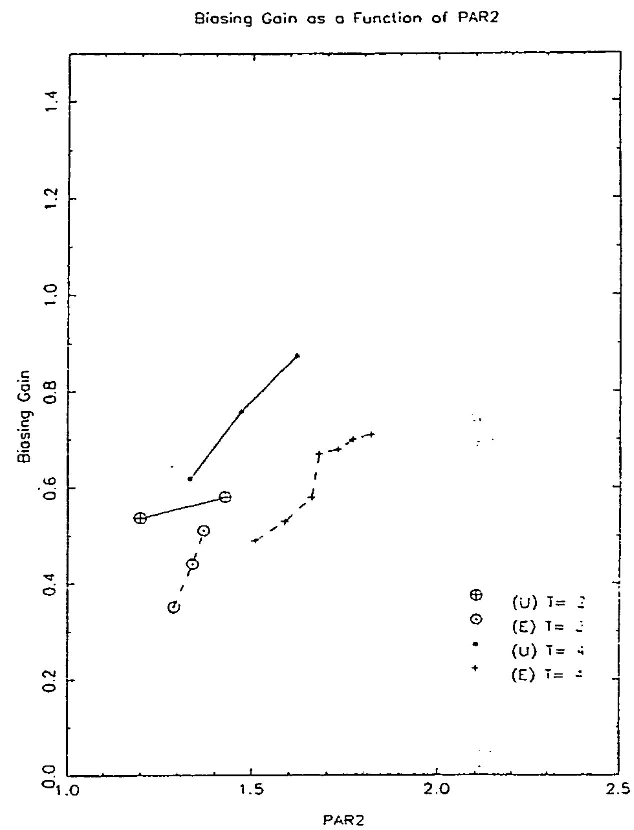

Fig. 4. Comparison of realizable biasing gains for both constructions.

However, using the new method, since the subconstellations each represent a different number of bits, errors on reception can lead to bit insertions and bit deletions. This is a significant problem, and must be addressed in any practical system. Here, we describe two proposed methods to deal with this, and propose an application where insertions and deletions do not cause unsolvable system problems.

The first approach we describe was proposed by Gallager in [12]. This document describes a system that implements shaping through the use of a secondary channel. The constellation is divided into subconstellations, similar to that as described in Section II. However, the extra bits needed to select points in the outer subconstellations are drawn from a secondary asynchronous source. The transmit sequence is designed in such a way that if bit deletions or insertions occur, the receiver automatically compensates by inserting or deleting, respectively, the same number of bits into the received asynchronous data stream. This ability to resynchronize, while not avoiding the erroneous reception of data, makes the approach attractive from a realizable system viewpoint.

The resynchronization is accomplished by use of a transmit buffer, and a specific schedule for the minimum number of bits to be held in the buffer at the end of each symbol transmission. If, after

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the end of a modulation period, there are not enough bits in the transmit buffer to meet the scheduled minimum, then the shortfall is made up from the secondary asynchronous data source. At the receiver, if an error is made such that it believes it has received, say, one more bit than was actually transmitted, then, because of the scheduled minimum number of bits in the transmit buffer, it will believe the extra bit came from the secondary data source, and send it to the secondary data sink, thus re-acquiring synchronization with the primary data source. A similar process occurs if the receiver believes one less bit has been transmitted than actually sent. In this case, the transmitter will at some time transmit an extra bit from the secondary data source, but the receiver will send it to the primary data sink, thus again resynchronizing the primary data. For a full description of this ingenious approach, the reader is referred to [12].

This approach points out one of the problems that variable-rate signaling faces, that of buffer management. Since a random number of bits are transmitted during each interval, some buffering must eventually be employed. During any transmission interval, the encoder may request either more or fewer bits than requested in the previous transmission interval. How this data is provided without the input buffer overflowing or becoming empty is of serious concern. Any system using variable-rate signaling will require some scheme to provide input data to the encoder in a timely fashion.

The second method proposed to combat this problem, is to partition the input data into constant length blocks, and then force the system to transmit this information using the same number of symbols each time. This method is described by Calderbank and Klimesh in [13], and uses what are known as balanced codes. Balanced codes have the property that every codeword has an equal number of zeros and ones. Using a simple construction provided by Knuth in [14], Calderbank and Klimesh are able to construct rate $2^p/(2^p + p)$ balanced codes with a particularly simple method of encoding and decoding. Using this approach on nonequal subconstellations, with codes of high rate, they are able to preserve almost all of their shaping gain. For example, it is possible to construct a rate 256/264 balanced code. Since the number of zeros and ones are equal in each word, we have $f_0 = f_1 = 1/2$. Using this code as the shaping code, with nonequal subconstellations having $k_0 = 1$ and $k_1 = 3$, yields a biasing gain of 0.44 dB, only 0.09 dB less than the biasing gain offered using a rate 1 code. In this case, we would have to partition the input data into blocks corresponding to 264 symbols, with half of the symbols drawn from the inner region. Thus every block would transmit $132\log_2(|\Omega_0|) + 132 \times 2\log_2(|\Omega_0|) + 256$ bits.

A place where the shaping described in Section II could be applied most simply is in an ARQ system. Here, the input data is collected into a group of a fixed number of bits to form a packet. A cyclic redundancy check code is applied during transmission. Upon reception, the CRC is used to check for errors. If errors have occurred, then a retransmission is requested. Headers and trailers using fixed constellations would also simplify this system, allowing the receiver to detect the starting and ending sequences that delineate a packet. Again, the insertion or deletion of bits in this case is not a fatal error, since a retransmission can solve the problem. However, if the system is using synchronous multiplexing between ports, then bit insertions and deletions may prove to be an insurmountable problem.

VI. A Design Example

We conclude this correspondence with a simple example of shaping using nonequal subconstellations. In this example, we select $T = 4$, with $k_1 = 3$, $k_2 = 7$, and $k_3 = 15$. We select as a fundamental region $\mathscr{R}$ the 16-QAM constellation, with the second, third,

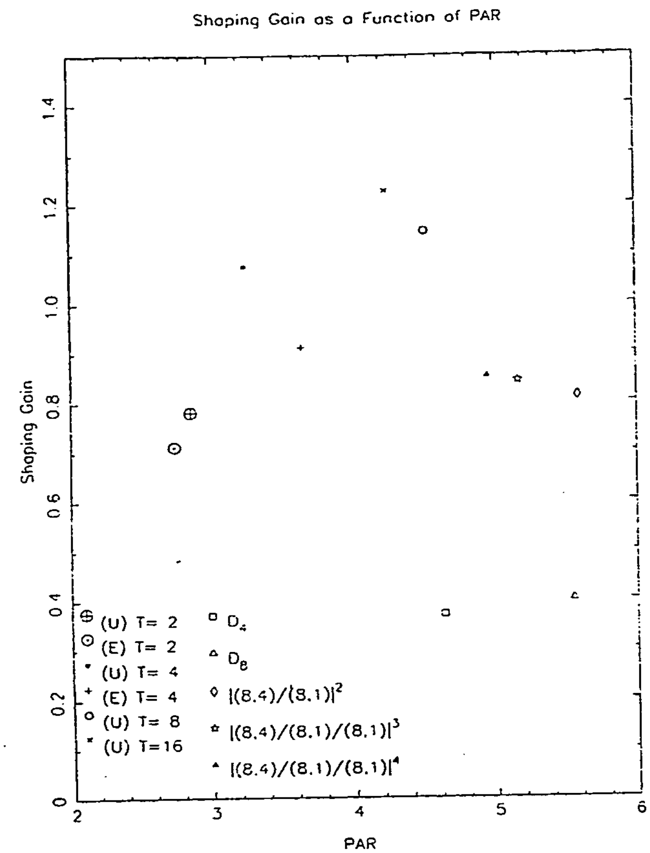

Fig. 5. Comparison between shaping gains offered by several different approaches.

and fourth subconstellations containing 32, 64, and 128 points, respectively. Thus, our total constellation will consist of the 240 points illustrated in Fig. 6. Note that the subconstellations are outlined in this figure.

We will use an equiprobable subconstellation selection shaping code. With four subconstellations, the rate is 2 bits/channel use. Thus the biasing gain is theoretically 0.87 dB. Calculating the average power of each subconstellation, we find that (assuming the points lie on an odd half-integer lattice) $P_{\Omega_0} = 2.5$, $P_{\Omega_1} = 10.25$, $P_{\Omega_2} = 25.375$, and $P_{\Omega_3} = 56.125$. Thus, the total average power is $P_s = 23.6$. Note that the constellation, as it stands, has a shaping gain of nearly 0.2 dB.

To compare for gain purposes, we note that given the constellation in Fig. 6, the average number of bits transmitted per channel use is 7.5 bits. Computing the baseline average power as in [2], we find $P_u = 2^{7.5}/6 = 30.17$. Thus, the total shaping gain (including biasing gain and constellation shaping gain) is $10\log_{10}(30.17/23.6) = 1.07$ dB, equaling the sum of the biasing gain and the shaping gain of the constellation. This is a significant gain for little complexity. If we were to use balanced shaping codes as proposed by Calderbank, then, using two rate 1024/1034 codes, the loss in biasing gain would be $2^{20/1034} = 0.06$ dB, yielding an overall gain of 1.01 dB.

Similarly, $\mathrm{PAR}_2 = 1.76$ and $\mathrm{CER} = 1.33$. A better comparison than PAR_2 is to use the ratio of the PAR_s of the shaped constellation to the PAR_u of the unshaped constellation. This yields

$$\frac{\mathrm{PAR}_s}{\mathrm{PAR}_u} = 1.22. \tag{18}$$

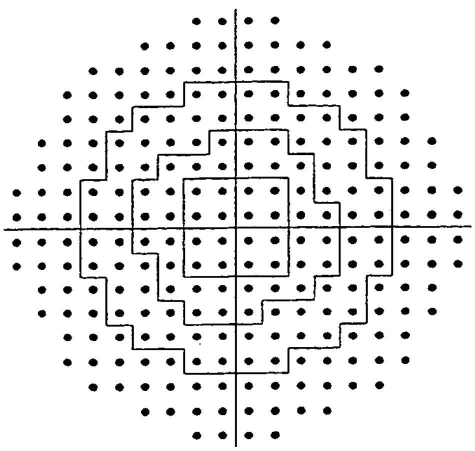

Fig. 6. 240 point constellation used in the design example to achieve 1 dB of shaping gain.

Thus, the increase in PAR is actually less than that indicated by PAR_2.

VII. Conclusion

We have described an approach to achieving shape gains that, while not completely new, has largely been ignored. We have demonstrated the power of the approach in that it achieves higher shaping gains than several other schemes, and with less constellation expansion and peak-to-average power increase. However, there are serious system issues to be resolved before the method can be successfully applied.

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Weight Distribution of Cosets of 2-Error-Correcting Binary BCH Codes of Length 15, 63, and 255

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Abstract—**The weight distributions of cosets are not known for the class of binary 2-error-correcting BCH codes of length $n = 2^m - 1$, m even (the nonuniformly packed case). By using the graph theoretical concepts of combinatorial matrix of a code and r-partition design introduced in previous works, we obtain these weight distributions of cosets in the three particular cases of length 15, 63, and 255. We observe that in these three cases the number of distinct weight distributions is a constant equal to 8.**

Index Terms—**BCH codes, weight distribution of cosets, automorphism groups, coherent partitions, partition designs, regularity number.**

I. Introduction

In [6], the notion of r-partition designs was introduced to extend to more general situations some properties of partial difference sets [3], [7] which is the particular case corresponding to 2-partition designs. If a linear code C admits a r-partition design, which means that the set of columns of a parity matrix of C is a class of the partition, then the theory developed in [6] gives the weights of the orthogonal code and allows us to effectively calculate the distance matrix of the code whose rows give the coset weight distributions of C. The aim of this paper is to use the methods of [6] to calculate the distance matrix in the three unknown cases of 2-error-correcting binary BCH codes of length 15, 63, and 255.

The class of binary 2-error-correcting BCH codes of length $n = 2^m - 1$ and codimension $2m$ has been extensively studied. In 1960, Gorenstein, Peterson, and Zierler [11] obtained that these codes are quasiperfect, the covering radius being 3. In 1967, Kasami [12] has determined the weight distributions of these codes. When m is odd, the orthogonal code has three nonzero weights whereas when m is even it has five nonzero weights. So when m is odd the binary 2-error-correcting BCH codes of length $n = 2^m - 1$ are uniformly packed [15] by a theorem of Goethals and Van Tilborg [10], [16] (see also [1] and [2]) and the theory of Van

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